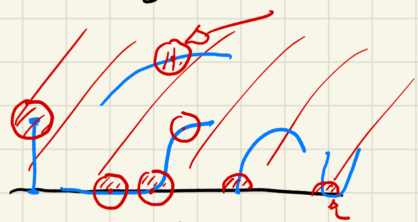
Varietà con bordo

$df_p : T_pM \to T_{f(p)}N$

Def: $f: M \to N$ M, N con bordo liscia, immersione, embedding come prima

Def: Una SOTTOVARIETÀ di una varietà con bordo N è l'immagine di un embedding $f: M \hookrightarrow N$.

M, N entrambe con bordo

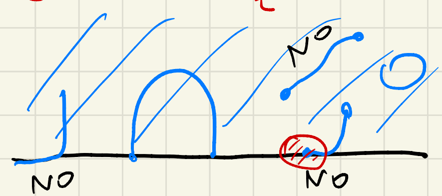


$N = \mathbb{R}^2_+$

La sottovarietà è NEAT se

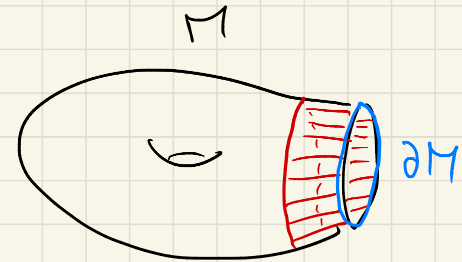
$$f(\partial M) = f(M) \cap \partial N$$

$\forall p \in \partial M$ $df_p(T_pM)$ TRASVERSO a $T_{f(p)}N$



OPERAZIONI CUT & PASTE

Incollamento lungo il bordo:



Def M varietà con bordo.

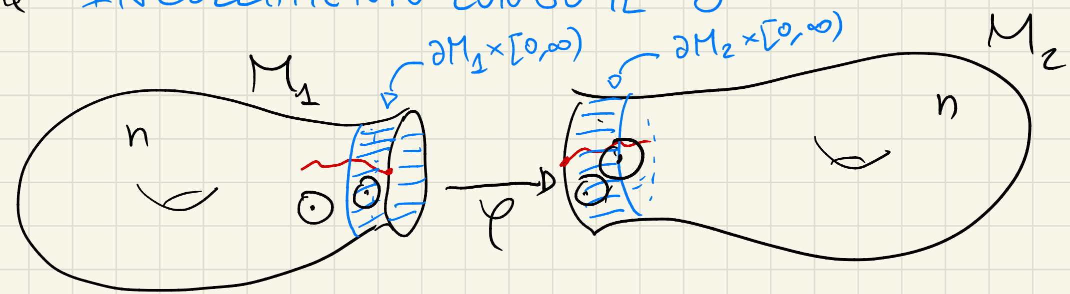
Un COLLARE per ∂M è

$$i : \partial M \times [0,\infty) \hookrightarrow M$$

t.c. $i(x,0) = x \quad \forall x \in \partial M$

Teo: Collare $\exists !$ a meno di isotopia

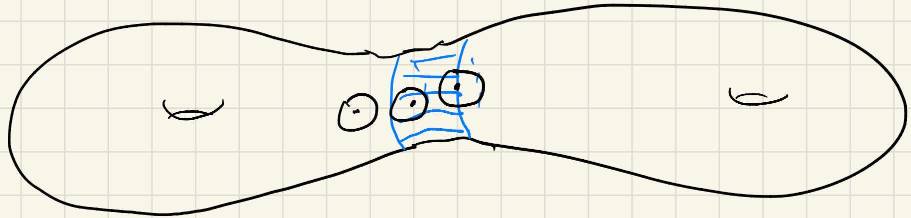
Operazione INCOLLAMENTO LUNGO IL ∂



ALTRA STRADA

$$\varphi : \partial M_1 \xrightarrow{\sim} \partial M_2 \text{ diffeo}$$

$$M = M_1 \sqcup M_2 /_{\sim} \qquad \boxed{x \sim \varphi(x)} \quad \forall x \in \partial M_1$$

$M =$ 

Si ottiene una varietà topologica

Quale atlante liscio diamo a M?

Nuova definizione:

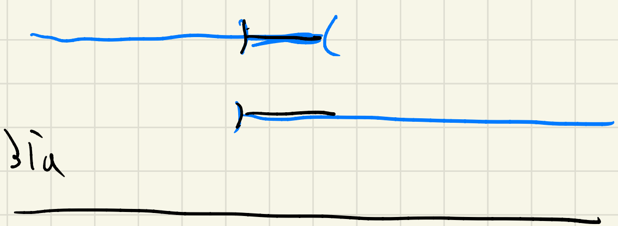
$$M = (M_1 \setminus \partial M_1) \sqcup (M_2 \setminus \partial M_2) /_{\sim} \qquad (x,t) \sim \left(\varphi(x), \tfrac{1}{t}\right)$$

$$\forall (x,t) \in \partial M_1 \times (0, \infty)$$

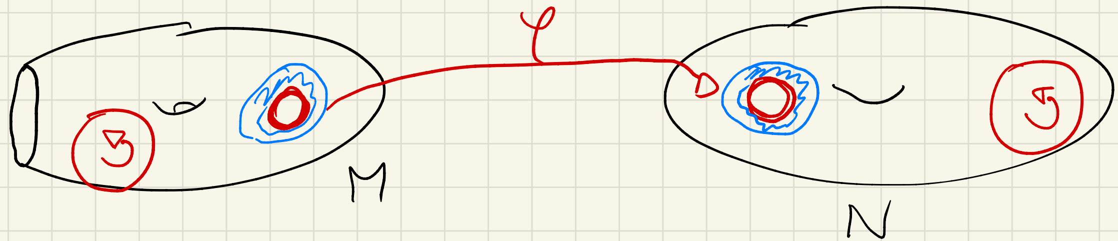
$\exists$ atlante naturale che estende quelli di M_1 e M_2

SOMMA CONNESSA

Teo: La nuova M non dipende (a meno di diffeo) dai collari scelti



Se cambiamo φ con una isotopia, M non cambia



M, N connesse $\dim M = \dim N = n$ orientate

$f : \mathbb{R}^n \hookrightarrow M$ pres. ori.

$g : \mathbb{R}^n \hookrightarrow N$ inverte ori.

Ricordiamo: f & g sono uniche a meno di isotopia

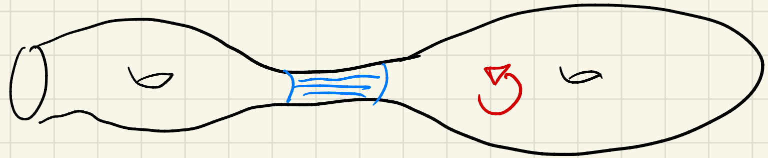
$$M \# N = \underbrace{M \setminus f(B^n)}_{M'} \qquad \underbrace{N \setminus g(B^n)}_{N'}$$

$$\partial M' = \partial M \cup \partial f(B^n) \overset{\shortparallel}{} f(S^{n-1})$$

$$\varphi := g \circ f^{-1}$$

$f : \mathbb{R}^2 \hookrightarrow$

$f' : \mathbb{R} \hookrightarrow$



$M \# N$

SOMMA CONNESSA DI M E N

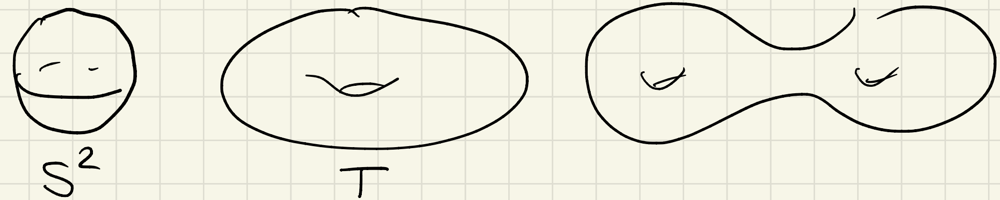
Non dipende da nulla (eccetto M e N)

L'operazione $\#$ sulle varietà di dim n è:

- $M \# N \cong N \# M$
- $(M \# N) \# P = M \# (N \# P)$
- $M \# S^n \cong M$

Superfici

cpt senza ∂
orientate



$\forall g \geq 1$ $\quad S_g := \underbrace{T \# \cdots \# T}_{g}$

TRASVERSALITÀ

$V \supseteq U, W$ sottospazi TRASVERSI se $U + W = V$

Def $f : M^m \to N^n$ $\quad g : W^w \to N^n$ lisce fra varietà

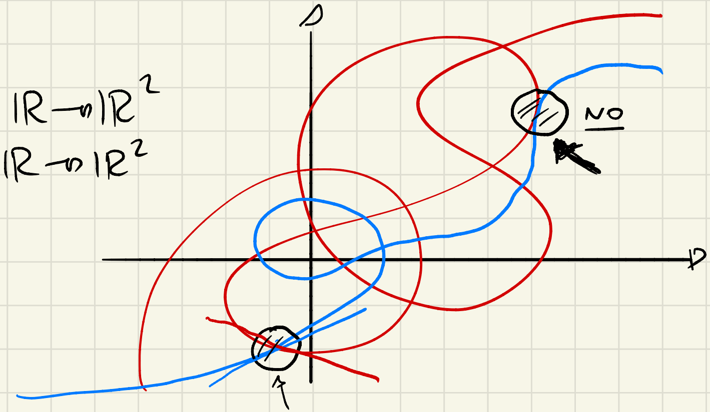
sono TASVERSE se $\forall p \in M, q \in W$ t.c. $f(p) = g(q)$

$f \pitchfork g$ allora $\boxed{\underset{\mathrm{Im}\, df_p}{\overset{\shortparallel}{df_p(T_pM)}} + \underset{\mathrm{Im}\, dg_q}{\overset{\shortparallel}{dg_q(T_qM)}} = T_{f(p)}N}$

Oss: Se $m+w<n$, allora $f \pitchfork g \iff \mathrm{Im} f \cap \mathrm{Im} g = \emptyset$

Def: Se $\left.\begin{array}{l} f: M \to N \\ W \subseteq N \end{array}\right\}$ TRASVERSE

se lo sono f e $i: W \hookrightarrow N$



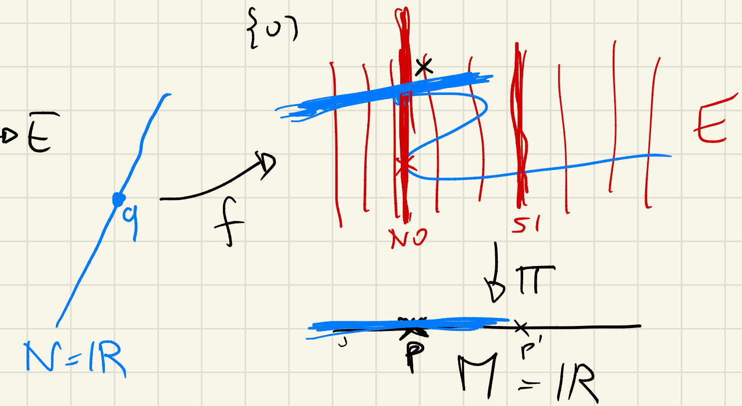
Def: $M, W \subseteq N$ TRASVERSE

se lo sono le inclusioni cioè $\forall p \in M \cap W$, $T_pM + T_pW = T_pN$

Oss: $f: M \to N$ $\quad W = \{p\} \in N$ $\quad f \pitchfork \{p\}$? $\iff$ p valore reg. per f

$\forall q \in M : f(q) = p \qquad \mathrm{Im}\, df_q + \underbrace{T_p P}_{\{0\}} = T_p N$

Prop: $\begin{array}{c} E \\ \downarrow \pi \\ M \end{array}$ fibrato $\qquad f : N \to E$



$\boxed{f \pitchfork E_p \iff p \text{ è val. reg. per } \pi \circ f}$

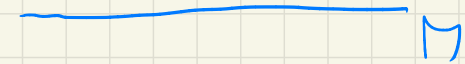
$\ker d\pi_x = T_x E_p$

Ex: $\begin{array}{c} E \\ \downarrow \pi \\ M \end{array} \supseteq W$ sottovarietà $\qquad W$ è immagine di una qualche sezione $s : M \to E \iff$ è trasversa a E_p $\forall p$ e interseca ogni E_p in 1 punto

W interseca trasv. ogni fibra in 1 punto

Prop: $M^m \subseteq N^n$ sottovarietà

$g : W^w \to N$ $\qquad g \pitchfork M$



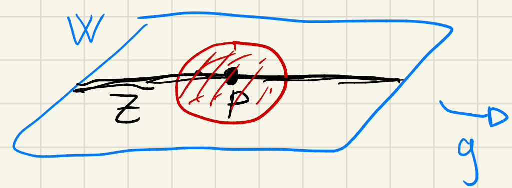
$\Rightarrow g^{-1}(M) \subseteq W$ sottovarietà di codim $n-m$

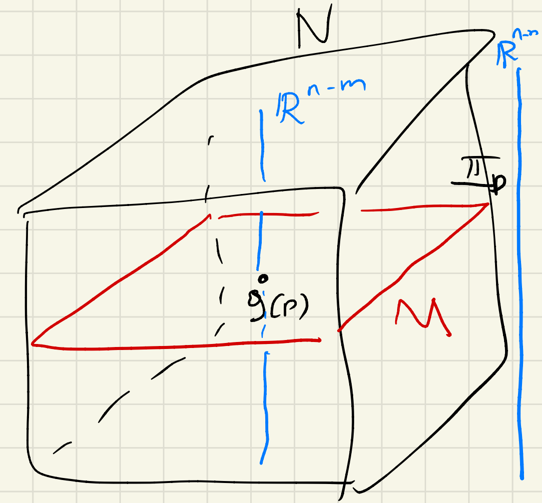
dim:

$Z = g^{-1}(M)$

Ts: $Z \subseteq W$ sottovarietà

$p \in Z$:





$Z = g^{-1}(M)$

$= g^{-1}\pi^{-1}(0)$

$= (\pi \circ g)^{-1}(0)$

$M = \mathbb{R}^m \subseteq \mathbb{R}^n$

$N = \mathbb{R}^n$

$\pi : \mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-n} \to \mathbb{R}^{n-m}$

FIBRATO

$M = \pi^{-1}(0)$

$g \pitchfork M \iff 0$ val. reg. per $\pi \circ g$

$\Rightarrow Z = (\pi \circ g)^{-1}(0)$ varietà

Cor: $M^m, W^w \subseteq N^n$ trasverse $\Rightarrow Z = M^m \cap W^w$ è varietà

$$\operatorname{codim} Z = \operatorname{codim} M + \operatorname{codim} W$$

Teo (TRASVERSALITÀ DI THOM)

$F: M \times S \to N$ liscia $\qquad Z \subset N$ sottovarietà

$F \pitchfork Z \qquad \forall s \in S: F_s : M \to N$
$\qquad\qquad\qquad\qquad p \mapsto F(p,s)$

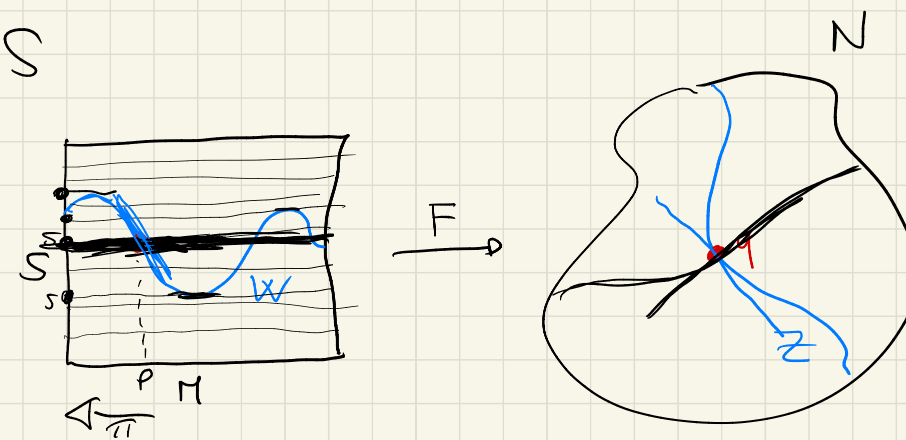
$\Downarrow$

$F_s \pitchfork Z$ per q.o. $s \in S$

dim:

$W = F^{-1}(Z) \subseteq M$
sottovarietà

$\pi: M \times S \to S$



$s \in S$ è val. reg. per $\pi|_W : W \to S \iff W \pitchfork M \times \{s\}$

$\underline{T_S}$: Se $s \in S$ è valore reg. per $\pi|_W$, allora $F_s \pitchfork Z$

$\underline{\text{Sard}}$: $s \in S$ val. reg. per $\pi|_W$ per q.o. $s \in S$

$\underline{Hp}$: $\underline{s}$ val. reg. per $\pi|_W \iff W \pitchfork M \times \{s\}$

$(p, s) \in W$ $\qquad$ $\underline{\text{Tesi}}$: $\underbrace{(dF_s)_p (T_p M)}_{= dF_{(p,s)}(T_{(p,s)}(M \times \{s\}))} + T_q Z = T_q N$

1 $\qquad$ 2 $\qquad$ 3

$\underline{Hp}$: $T_{(p,s)} W + T_{(p,s)} M \times \{s\} = T_{(p,s)}(M \times S)$ Applico $dF_{(p,s)}$

$$\underbrace{dF_{(p,s)}(T_{(p,s)} W)}_{= T_q Z} + dF_{(p,s)}(T_{(p,s)} M \times \{s\}) = dF_{(p,s)} T_{(p,s)}(M \times S)$$

2 $\qquad$ 1 $\qquad$ 3

$\square$

$F: M \times S \to N \supseteq Z$ Se $F \pitchfork Z$ allora $F_s \pitchfork Z$ per q.o. $s \in S$

Cor: $f: M \to \mathbb{R}^n \supseteq Z$

Per q.o. $s \in \mathbb{R}^n$ $f_s: M \to \mathbb{R}^n$ $\pitchfork Z$
$p \mapsto f(p) + s$

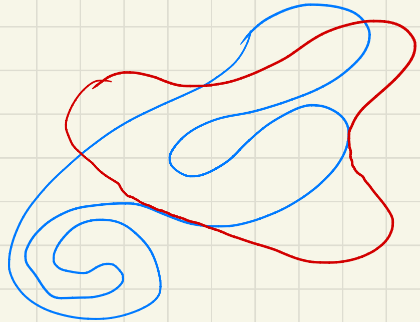
dim:

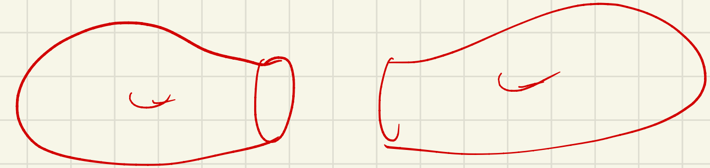
$F: M \times \mathbb{R}^n \to \mathbb{R}^n$
$(p, s) \longmapsto f_s(p) = f(p) + s$

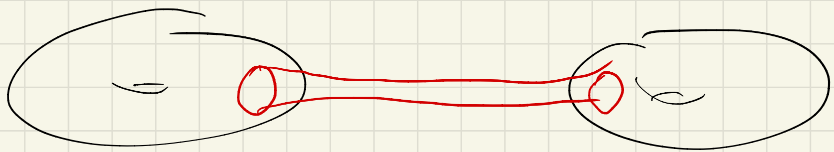
$F \pitchfork Z$? dF suriettivo ovunque F sommersione

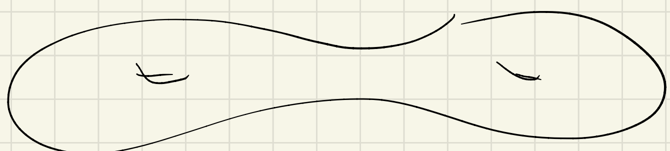
$\Rightarrow$ TESI. (applico Thom) $\square$

Cor: $M, Z \subseteq \mathbb{R}^n$ Per q.o. $s \in \mathbb{R}^n$ $M_s = M + s$ è trasv. a Z









NO

$n \geq 3$

$\pi_1(M \# N) = \pi_1(M) * \pi_1(N)$

$C^\infty = \{f : M \to N\}$

Essere X è stabile

trasversi

$f: M \to N$ $\quad Z \subseteq N$

$f \pitchfork Z$

$f_t : M \to N$

$t = 0 \Rightarrow \forall t < \varepsilon$